Young Tableaux and Probability

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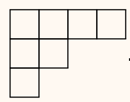
1 Background and motivation

Definition 1.1

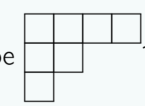
A **Young diagram** of size n is a collection of n left-justified boxes in rows:



A **standard Young tableau** (SYT) is a way to fill in the n boxes of a Young diagram with the numbers $1, 2, \cdots, n$ (without repetition) such that rows and columns are increasing towards the right and down. For example, a SYT of the Young diagram $(4, 2, 1)$ is

1	2	6	7
3	4		
5			

Example 1.2

How many standard Young Tableaux exist for the shape  ?

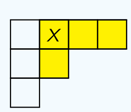
One way to calculate this is by doing casework on where the number 7 goes: since nothing is larger than 7 in this Young tableau, 7 must go in one of the bottom right corners. Now we can just find the number of standard Young tableaux of the smaller shapes, and repeat until we have our answer. Another way to calculate this is by starting from the upper-left corner and doing casework on where the next number can go. The casework could be very complicated, but a general yet elegant formula exists, for which mathematicians over generations have come up with many beautiful proofs.

Theorem 1.3 (Hook Length Formula)

The number of ways to fill out a standard Young tableau λ with n boxes is

$$f^\lambda = \frac{n!}{H(\lambda)} = \frac{n!}{\prod_{x \in \lambda} h(x)}$$

where $h(x)$ is the "hook length" of x. For example, in the below diagram, the hook length $h(x) = 4$.

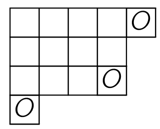


Shiyue gave a proof of this using generating functions and algebra in Week 2, and the goal of our class here is to give more of a reason for where the "hooks" come from!

2 The beginning of a proof

This hook walk proof originates from Greene, Nijenhuis, and Wilf (1979).

As stated earlier, there is a recurrence relation for the number of standard Young tableaux. Given a tableau with n boxes, n must appear in one of the bottom-right corner boxes, and removing this corner, we get a standard Young tableau of size $n-1$. For instance, look at the following:



Since the O's are the spots that the largest number can be in,

$$f^{5,4,4,1} = f^{4,4,4,1} + f^{5,4,3,1} + f^{5,4,4}.$$

So we'll try to prove the theorem by induction:

$$f^{\lambda} = \sum_{v \text{ corner of } \lambda} f^{\lambda - v}.$$

The base case, where we have 0 or 1 boxes, can be verified pretty easily: $f^{\varnothing} = 1$ and $f^{1} = 1$.

For the inductive step, now we need to show that the same recurrence relation holds for the expression on the right hand side! In other words, we want to show that

$$\frac{n!}{H(\lambda)} = \sum_{v \text{ corner of } \lambda} \frac{(n-1)!}{H(\lambda - v)}.$$

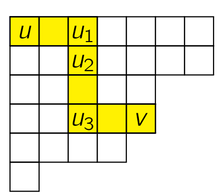
This is the same as wanting to show that (dividing through by the left hand side)

$$1 = \sum_{v \text{ corner}} \frac{1}{n} \frac{H(\lambda)}{H(\lambda - v)}.$$

We have a 1 on the left side, and we have a bunch of nonnegative real numbers on the right side. So our goal is to make this a probability question! In other words, we want to show that the probability of something related to a corner v of λ is

$$\frac{H(\lambda)}{nH(\lambda - v)}.$$

Let's say we start at a box u of our Young tableau – we'll describe what u is later on. At each step, jump from u to any other square in the hook of u with equal probability. Repeat this process repeatedly, and stop once we reach a corner v.



This is a **hook walk** from u to v.

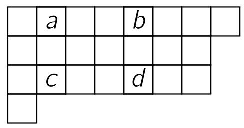
Denote $P(u, v)$ to be the probability that a hook walk starting at box u ends at corner v. If a hook has $h(x)$ total boxes, we move to any of the $h(x) - 1$ other boxes with equal probability:

$$P(u, v) = \sum_{u \to u' \to \cdots \to v} P(\text{this hook walk}) = \sum_{u \to u' \to \cdots \to v} \left(\frac{1}{h(u) - 1} \cdot \frac{1}{h(u') - 1} \cdots \right).$$

This seems a bit hard to calculate, so we'll break it up into more manageable chunks.

3 Changing perspectives: some algebraic manipulation

Here's the key observation: if we fix our ending box v, the whole hook-walk stays within the rectangle between the top-left corner and v. Looking inside this rectangle, whenever we have a smaller rectangle with corners a, b, c, d, such as in the below picture,



we have a relation

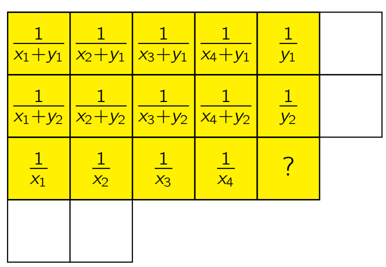
$$h(a) + h(d) = h(b) + h(c) \implies \boxed{(h(a) - 1) + (h(d) - 1) = (h(b) - 1) + (h(c) - 1)}.$$

(In particular, if d is a corner, $h(d) - 1 = 0$, so this simplifies very nicely!)

Fact 3.1

Notice that with this rectangle law, we can uniquely determine the labels everywhere in our rectangle as long as we know the labels on the bottom and right edges of the rectangle.

Let's say the labels of the last row of our rectangle are $\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_k}, ?$, and the labels in the last column are $\frac{1}{y_1}, \frac{1}{y_2}, \cdots, \frac{1}{y_\ell}, ?$. Then our picture looks something like this:



(Note that the ? probably breaks rule 4, except that there are two staff here.)

Our goal from here is to calculate the total probability of all hook-walks, but let's start by calculating the sum of probabilities of all **lattice paths**: that is, hook-walks where we can only move down or right by one square each time.

Proposition 3.2

Given a Young diagram λ, let A, B be the upper-left and lower-right corners of a $(k+1) \times (\ell+1)$ rectangle in λ. Suppose the bottom row of the rectangle is labeled $\frac{1}{x_1}, \ldots, \frac{1}{x_k}$ and the rightmost column of the rectangle is labeled $\frac{1}{y_1}, \ldots, \frac{1}{y_\ell}$. The total probability of all lattice paths from A to B is

$$\frac{1}{x_1 x_2 \cdots x_k y_1 \cdots y_\ell}.$$

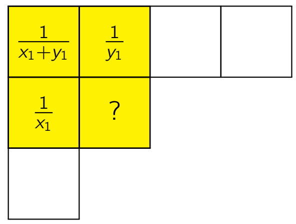
For example, for a 2 by 2 grid, the boxes are labeled as follows.

$\frac{1}{x_1+y_1}$	$\frac{1}{y_1}$
$\frac{1}{x_1}$	?

Then the total probability of all lattice paths is

$$\frac{1}{x_1+y_1}\cdot\frac{1}{y_1}+\frac{1}{x_1+y_1}\cdot\frac{1}{x_1}=\frac{1}{x_1y_1}.$$

Proof. We induct on $k+\ell$ (the sum of the length and width of our rectangle). Base case is $k+\ell=2$:



and indeed,

$$\left(\frac{1}{x_1+y_1}\cdot\frac{1}{x_1}\right)+\left(\frac{1}{x_1+y_1}\cdot\frac{1}{y_1}\right)=\frac{1}{x_1y_1}.$$

For the inductive step, if we start at rectangle corner A, which is labeled $\frac{1}{x_1+y_1}$, any lattice path either goes down one square from A (to A') or right from A (to A''). Thus the sum over all paths of $w(P)$ is

$$\frac{1}{x_1+y_1}\left(\sum_{P':A'\to B}w(P')+\sum_{P'':A''\to B}w(P'')\right).$$

However, by the inductive hypothesis, this is just

$$\frac{1}{x_1+y_1}\left(\frac{1}{x_1x_2\cdots x_ky_2\cdots y_\ell}+\frac{1}{x_2\cdots x_ky_1y_2\cdots y_\ell}\right)$$

which simplifies to

$$\frac{1}{x_2\cdots x_ky_2\cdots y_\ell}\cdot\frac{1}{x_1+y_1}\left(\frac{1}{x_1}+\frac{1}{y_1}\right)=\frac{1}{x_1\cdots x_ky_1\cdots y_\ell},$$

as desired. $\square$

Try this exercise to gain some more intuition for why we are proving the hook-length formula using induction! (The proof for the problem below is much simpler, though.)

Exercise 3.3

Prove that in a rooted tree T (with root at top) with n vertices, the number of ways to label the vertices 1 through n so that all the labels are smaller than the parents' labels is

$$\frac{n!}{\prod_{v\in T}h(v)}$$

where $h(v)$ is the number of vertices in the subtree originating from v (including v).

4 Review

Yesterday, we started discussing the Hook Length Formula:

Theorem 4.1 (Hook Length Formula)

The number of ways to fill out a standard Young tableau λ with n boxes is

$$f^\lambda = \frac{n!}{H(\lambda)} = \frac{n!}{\prod_{x \in \lambda} h(x)}$$

where $h(x)$ is the "hook length" of x.

Using the intuition that we are filling out our numbers in increasing order until we reach the largest value n, we sought to prove our formula inductively: this led us to consider the quantity

$$1 \stackrel{?}{=} \sum_{v \text{ corners}} \frac{H(\lambda)}{nH(\lambda - v)},$$

where $H(\lambda)$ is the product of the hook lengths in the entire Young diagram. This motivated us to try to use a probabilistic **hook-walk** to interpret the right hand side.

Remember that if we fix the start and end points, we are constrained to a rectangle of movement. We label each box in the rectangle with the probability of moving from this box to any of its hook members. Last time, our restricted rectangle of a $(k+1)$ by $(\ell+1)$ grid assigned each box a label with the following rules:

- The bottom row has labels $\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_k}, 1$.
- The right column has labels $\frac{1}{y_1}, \frac{1}{y_2}, \cdots, \frac{1}{y_\ell}, 1$.
- Any other grid square in the ith column and jth row has labels $\frac{1}{x_i + y_j}$, where $i \leq k$ and $j \leq \ell$.

We wanted to consider all lattice paths from A (top left corner) to B (bottom right corner), where each path P has a probability equal to the product of its labels.

Lemma 4.2

The total probability of all lattice paths

$$\sum_{P: A \to B} w(P) = \frac{1}{x_1 x_2 \cdots x_k y_1 y_2 \cdots y_\ell}.$$

We proved this using induction last time, and now we are finally able to connect it back to the hook-length formula!

5 Working with our lemma and getting to the punchline

In the definition of our hook walk, we're allowed to jump across multiple rows and columns to a hook member, so this is not exactly what we need in our proof. Recall the exact definition of hook walk: we start at some vertex u in our grid graph, and we repeatedly jump to a vertex that is either to the right of or below our current spot, but we always end at some bottom-right corner v of our Young diagram. Also recall that the definition of lattice path is a hook walk where every movement is an adjacent one.

Lemma 5.1

If we replace "lattice path" with "hook-walk" in the example above,

$$\sum_{P \text{ hook-walk}} w(P) = \left(1+\frac{1}{x_1}\right)\left(1+\frac{1}{x_2}\right)\cdots\left(1+\frac{1}{x_k}\right)\left(1+\frac{1}{y_1}\right)\cdots\left(1+\frac{1}{y_\ell}\right).$$

Proof. Expand the product! Each term corresponds to a subgrid: we skip certain rows and columns if we use a 1 instead of a $\frac{1}{x_i}$ or $\frac{1}{y_j}$. Recall that we do not need to start at the top left corner, so we do not need to necessarily include $\frac{1}{x_1}$ and $\frac{1}{y_1}$. Also, the factor of $1 \cdot 1 \cdot \cdots \cdot 1$ just starts at B and finishes immediately. □

So now it's time to go back to notions from the first day of class. We denote $P(u, v)$ to be the probability that we end up at corner box v, given that we start at u. Any path that ends up at v stays in the rectangular grid between the top left corner and v. This can be written as

$$= \sum_{p:u\to v} \frac{1}{h(u)-1} \cdot \frac{1}{h(u')-1} \cdot \frac{1}{h(u'')-1} \cdots .$$

Remember that these denominators are exactly the $x_i + y_j$ terms that we saw in our lemmas! In particular, if we apply the second lemma on a particular corner box v, calling the set of boxes above or to the left of v the "cohook of v,"

$$\sum_u P(u, v) = \prod_{t \text{ in cohook } v} \left(1 + \frac{1}{h(t)-1}\right) = \prod_t \frac{h(t)}{h(t)-1}$$

But now this is easily simplified. If we remove v from the hook-length, what happens to the product of hook lengths $H(\lambda)$? Nothing, except all hook lengths in the co-hook decrease by one! So this is just equal to

$$\sum_u P(u, v) = \frac{H(\lambda)}{H(\lambda - v)}.$$

So it's time for the punchline: for any fixed box u of λ,

$$\sum_{v \text{ corner}} P(u, v) = 1,$$

since any hook walk ends at some corner. Here's the final key step: adding over all $u \in \lambda$ and all corner boxes v,

$$\sum_{u\in\lambda} \sum_{v \text{ corner}} P(u, v) = n,$$

since there are n boxes in our Young tableaux. But now switch the order of summation:

$$\sum_{v \text{ corner}} \sum_{u\in\lambda} P(u, v) = n,$$

and thus by substituting in what we had above,

$$\sum_{v \text{ corner}} \frac{H(\lambda)}{H(\lambda - v)} = n.$$

But now we can multiply both sides by $(n-1)!$ and rearrange:

$$\frac{n!}{H(\lambda)} = \sum_{v \text{ corner}} \frac{(n-1)!}{H(\lambda - v)}$$

which is exactly the recurrence relation we were trying to prove, and we're done!

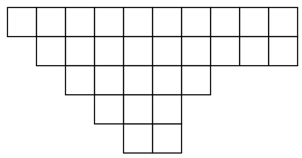
6 So what's next?

The remainder of this class will take this idea introduced in the 1979 paper and see how it was generalized and reapplied to subsequent ideas after that.

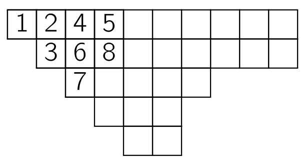
First of all, this process can be used to **generate random Young tableaux**. If we want to pick a random Young tableau for a given Young diagram (or shape) λ, we already know the probability that n is going to be in each corner box: it's equal to that quantity $\frac{1}{n}\frac{H(\lambda)}{H(\lambda - v)}$ from before! So we can carry this out recursively and end up with a uniformly random Young tableau.

6.1 Shifted Young Tableaux

Now let's talk about the concept of "shifted Young diagrams" – this connection was made by Bruce Sagan. Instead of left-justifying, we can have diagrams like



coresponding to a partition $\lambda = (10, 9, 5, 3, 2)$. This now gives ways to partition a number n into **distinct** parts, just like standard Young diagrams let us partition n in any way. To fill in our shifted Young tableau, we have the same condition – numbers still increase by row and by column:

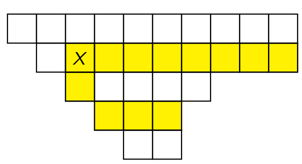


Theorem 6.1 (Thrall)
The number of shifted Young tableaux with shifted shape λ with n boxes is similarly

$$\frac{n!}{\prod_{a \in \lambda} h(a)}$$

where $h(a)$, the hook length, now includes a "broken leg."

For example, here's a hook with a broken leg:



Basically, if the hook reaches the left staircase (not the bottom or the right part), it bends over and continues. The proof here follows a very similar idea to our original hook length proof: we show that the randomly-generated Young tableaux all come up with equal probability! The proof is an entire paper three times as long as the original one, but there are two central ideas that are the same:

- We want to start from a random box of our Young diagram and keep doing our hook walk until we run into a corner.

- Terms that look like
$$\frac{1}{x_1x_2\cdots x_ky_1y_2\cdots y_\ell z_1\cdots z_m}$$
come up, corresponding to the left, bottom, and right corners of the rectangle analog in our earlier proof.

Things are much less clear though, and lots of grouping and casework are involved. Read the paper if you're interested in 1979 algebraic manipulations that take pages and pages to resolve!

6.2 Modifying the hook walk

A second paper by the original three authors (Greene, Nijenhuis, and Wilf) in 1981 uses a slightly different probabilistic approach on Young tableaux to prove the identity

$$\sum_{\lambda n \text{boxes}} (f^\lambda)^2 = n!$$

This can be proved with the Schensted correspondence or with representation theory, and there are great sources on this in various online places. But we can use a similar idea as above: if we could generate random Young tableaux each with probability $\frac{H(\lambda)}{n!}$ to find that there are $\frac{n!}{H(\lambda)}$ total tableaux, we can generate random Young tableaux of **any shape** with n boxes, such that each Young tableau appears with probability $\frac{f^\lambda}{n!}$. The result then follows, because each Young diagram appears with probability $\frac{(f^\lambda)^2}{n!}$.

Once again, the idea is to induct! We wish to come up with a random process of generating Young tableaux such that
$$\Pr(\lambda) = \frac{(f^\lambda)^2}{n!} \stackrel{?}{=} \sum_{\lambda' \to \lambda} \Pr(\lambda' \to \lambda)\frac{(f^{\lambda'})^2}{(n-1)!},$$
where we sum over all λ' which are the result of removing a corner box from λ. Then our goal is, not surprisingly, to find a process that
$$\Pr(\lambda' \to \lambda) = \frac{f^\lambda}{nf^{\lambda'}},$$
which should look very familiar. Well, we can construct a process that does this, and all that remains is to show that these can actually be probabilities: that
$$\sum_\lambda f(\lambda) = nf(\lambda'),$$
summing over all λ obtained from λ'. Try this out if you want to have some fun!